3.2: LINEARITY & WRONSKIAN

Theorem If $y = y_1(t)$ AND $y = y_2(t)$ ARE TWO SOL'NS TO $ay'' + by' + cy \stackrel{?}{=} 0$, THEN

$$y = c_1 y_1(t) + c_2 y_2(t) \text{ IS ALSO A SOL'N.}$$

proof GIVEN
$$a y_1'' + b y_1' + c y_1 = 0$$
$$a y_2'' + b y_2' + c y_2 = 0$$

THEN
$$a(c_1 y_1'' + c_2 y_2'') + b(c_1 y_1' + c_2 y_2') + c(c_1 y_1 + c_2 y_2)$$
$$= c_1(a y_1'' + b y_1' + c y_1) + c_2(a y_2'' + b y_2' + c y_2)$$
$$= c_1 \cdot 0 + c_2 \cdot 0 = 0 \; //$$

Ex $y'' + y' - 6y = 0$ GUESS $y = e^{rt}$

$$e^{rt}(r^2 + r - 6) \stackrel{?}{=} 0 \quad \text{CHARACTERISTIC EQUATION}$$
$$e^{rt}(r+3)(r-2) \stackrel{?}{=} 0$$
$$r_1 = -3, \; r_2 = 2$$

$$\Rightarrow y_1 = e^{-3t}, \; y_2 = e^{2t}$$

ANY LINEAR COMBINATION OF THIS TWO IS ALSO A SOL'N!!!

Here are two random #'s $c_1 = 4$, $c_2 = 7$

$\Rightarrow y = 4e^{-3t} + 7e^{2t}$ IS ALSO A SOL'N

CHECK!!

INITIAL CONDITIONS - THEORY

If $y = c_1 y_1(t) + c_2 y_2(t)$, $\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$

$$\Rightarrow \quad \begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

Examples of 2-by-2 systems:

$\boxed{A}$ $c_1 + 2c_2 = 10$
$c_1 - c_2 = 4$
UNIQUE SOL'N
$c_1 = 6$, $c_2 = 2$

$\boxed{B}$ $7c_1 - c_2 = 8$
$-14c_1 + 2c_2 = 10$
NO SOL'NS!!!
CAN'T HAPPEN!
$7c_1 - c_2 = -5$

FACT:

$$\begin{aligned} ac_1 + bc_2 &= \alpha \\ cc_1 + dc_2 &= \beta \end{aligned}$$

WILL ALWAYS HAVE A UNIQUE SOL'N IF

$$ad - bc \neq 0 \qquad ad - bc = \underbrace{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}_{\text{THE DETERMINANT}}$$

Ex) $\boxed{A}$ $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -1 - 2 = -3 \neq 0$

$\boxed{B}$ $\begin{vmatrix} 7 & -1 \\ -14 & 2 \end{vmatrix} = 14 - 14 = 0$

ASIDE: CRAMER'S METHOD PUT DESIRED OUTPUT HERE

$$c_1 = \frac{\begin{vmatrix} 10 & 2 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{-18}{-3} = 6 \checkmark \qquad c_2 = \frac{\begin{vmatrix} 1 & 10 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{-6}{-3} = 2 \checkmark$$

Ex) $y'' + y' - 6y = 0$, $r_1 = -3$, $y_1 = e^{-3t}$
$r_2 = 2$, $y_2 = e^{2t}$

$y(0) = 3$, $y'(0) = 4$ $\qquad y = c_1 e^{-3t} + c_2 e^{2t}$

FIND c_1, c_2

$$c_1 e^0 + c_2 e^0 \overset{?}{=} 3 \Rightarrow c_1 + c_2 = 3$$
$$-3c_1 e^0 + 2c_2 e^0 \overset{?}{=} 4 \Rightarrow -3c_1 + 2c_2 = 4$$

$$c_1 = \frac{\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix}} = \frac{2}{5} \qquad c_2 = \frac{\begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix}} = \frac{13}{5}$$

$\longleftarrow$ NOT ZERO $\longrightarrow$

$$\boxed{y = \frac{2}{5} e^{-3t} + \frac{13}{5} e^{2t}}$$

CHECK: $y(0) = \frac{2}{5} + \frac{13}{5} = 3$ ✓

$y'(0) = \frac{-6}{5} + \frac{26}{5} = 4$ ✓

IN GENERAL $\qquad$ WRONSKIAN

$$\begin{vmatrix} e^{-3t_0} & e^{2t_0} \\ -3e^{-3t_0} & 2e^{2t_0} \end{vmatrix} = 2e^{-t_0} - 3e^{-t_0} = -e^{-t_0} \underbrace{\neq 0}_{\text{NEVER ZERO!}}$$

WE SAY $y_1 = e^{-3t}$, $y_2 = e^{2t}$
FORM A FUNDAMENTAL SET OF SOL'NS

AND THERE IS ALWAYS A UNIQUE WAY TO FIND

$$y = c_1 e^{-3t} + c_2 e^{2t}$$

$$y_1 = e^{-3t} \qquad y_2 = 7e^{-3t}$$

ARE ALSO TWO SOL'NS, BUT THEY DO NOT FORM A FUNDAMENTAL SET OF SOL'NS

$$\begin{vmatrix} e^{-3t} & 7e^{-3t} \\ -3e^{-3t} & -21e^{-3t} \end{vmatrix} = -21 - -21 = 0$$

NO!

CONCLUSIONS: GIVEN $ay'' + by' + cy = 0$

1. GOAL IS TO FIND TWO SOL'NS $y = y_1(t)$ AND $y = y_2(t)$

   SUCH THAT $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

   "TWO "LINEARLY INDEPENDENT SOL'NS"

2. IF WE CAN DO IT, THEN WE CAN FIND A UNIQUE SOL'N IN THE FORM

$$y = c_1 y_1(t) + c_2 y_2(t).$$

Ex) $y'' + 9y \stackrel{?}{=} 0$ (3.3)

$e^{rt}(r^2 + 9) \stackrel{?}{=} 0$

$r^2 = -9 \Rightarrow r = \pm 3i$

TURNS OUT $y_1(t) = \cos(3t)$
$y_2(t) = \sin(3t)$
ARE BOTH SOL'NS (CHECK!)

AND $\begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\cos^2(3t) + 3\sin^2(3t)$
$= 3 \neq 0$

THEY ARE LINEARLY INDEPENDENT!

$\Rightarrow$ General sol'n is $y = c_1\cos(3t) + c_2\sin(3t)$

Ex) $y'' + 4y' + 4y = 0$ (3.4)

$e^{rt}(r^2 + 4r + 4) = 0 \Rightarrow e^{rt}(r+2)^2 \stackrel{?}{=} 0$

$r = -2$ IS ONLY ONE (REPEATED) ROOT

$y_1 = e^{-2t}$ IS ONE SOL'N WE NEED ANOTHER?

WE WILL SEE HOW IN 3.4